Holomorphic Dynamics - Lecture 10

König's linearization of attracting fixed points

Thm Let $f : S \to S$ a holo map with

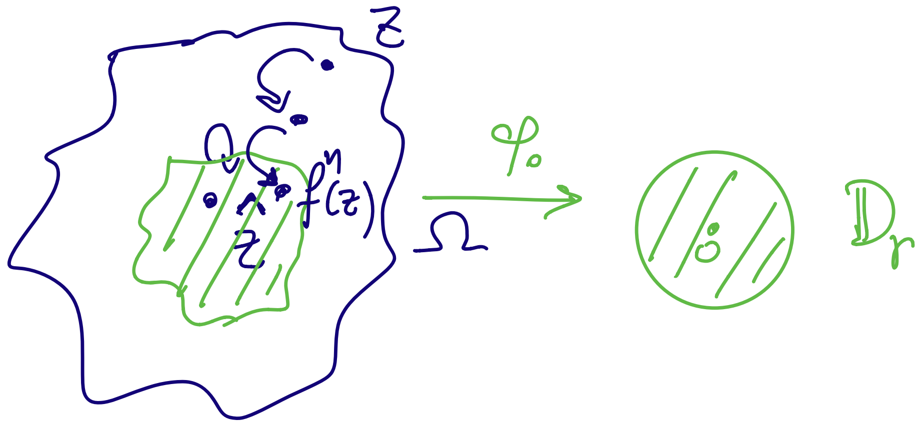
$f(\hat{z}) = \hat{z}$, $\quad \lambda = |f'(\hat{z})| < 1$.

Let $\Omega \subset S$ the basin of attraction of $\hat{z}$. Then there exists a holo $\varphi : \Omega \to \mathbb{C}$ s.t.

$$\begin{tikzcd} \Omega \arrow[r, "f"] \arrow[d, "\varphi"'] & \Omega \arrow[d, "\varphi"] \\ \mathbb{C} \arrow[r, "z \mapsto \lambda z"] & \mathbb{C} \end{tikzcd}$$

is commutative. Moreover, φ is unique up to multiplication by a constant.

Pf: Define $\varphi(z) = \dfrac{\varphi_0(f^n(z))}{\lambda^n}$



where φ_0 is the Königs map and
$n = \min\{K : f^K(z) \in \varphi_0^{-1}(\mathbb{D}_r)\}$.

Note.: φ is not injective, usually.

Thm (Fatou, Julia)

If f is a rational map of degree ≥ 2, then the immediate basin of any attracting fixed point contains a critical point. More precisely, if $\lambda \neq 0$, then there exists a unique compact nbd $\overline{\mathcal{U}}$ of $\hat{z}$ s.t.

1. $\overline{\mathcal{U}}$ maps bijectively to some $\overline{\mathbb{D}_r}$ under the Königs map
2. $\partial\mathcal{U}$ contains at least one critical point.

Cor.: The number of attracting cycles is $\leq 2d-2$ (= number of critical points)

$$f(z) = \frac{p(z)}{q(z)} \qquad f'(z) = \frac{p'q - pq'}{q^2}$$

Examples If $f(z) = z^2 + c$, then there is at most one attracting cycle (excluding $z = \infty$)

Proof

$$\begin{array}{ccc} \Omega & \xrightarrow{\varphi} & \mathbb{C} \\ \rotatebox{90}{$\in$} & \xleftarrow{\varphi^{-1}} & \rotatebox{90}{$\in$} \\ \hat{z} & & 0 \end{array}$$

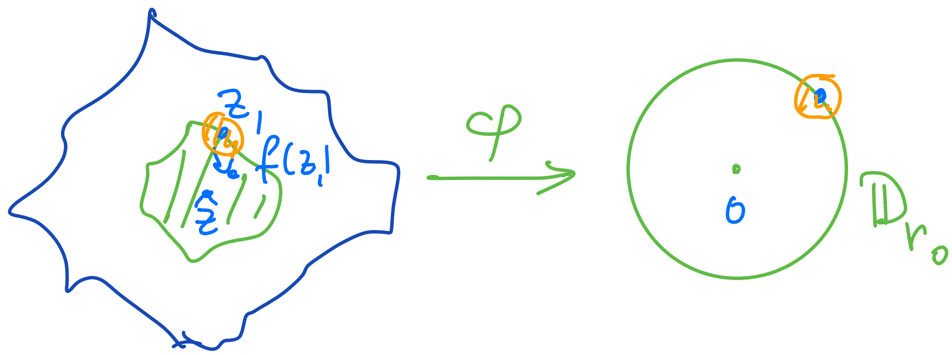
There exists (by Königs) some $r>0$ s.t. φ^{-1} is defined on D_r, let

$r_0 = \sup\{r>0 \text{ s.t. } \varphi^{-1}|_{D_r} \text{ is defined}\}$

We claim that

$r_0 < +\infty$.

Why? otherwise $\varphi^{-1} : \mathbb{C} \to \Omega$ must be constant by Liouville.



$\mathcal{U} := \varphi^{-1}(\mathbb{D}_{r_0})$. Let $z_1 \in \partial\mathcal{U}$.
Then $f(z_1) \in \mathcal{U}$, hence z_1 belongs to Ω (the basin)

We claim that at least one $z_1 \in \partial\mathcal{U}$ is critical. This is because otherwise you could extend φ^{-1} to a nbd of $\mathbb{D}_{r_0}$, hence r_0 would NOT be maximal.

Local Theory of Parabolic Points

$f(z) = \lambda z + a_2 z^2 + \cdots$ $\qquad f(0) = 0$

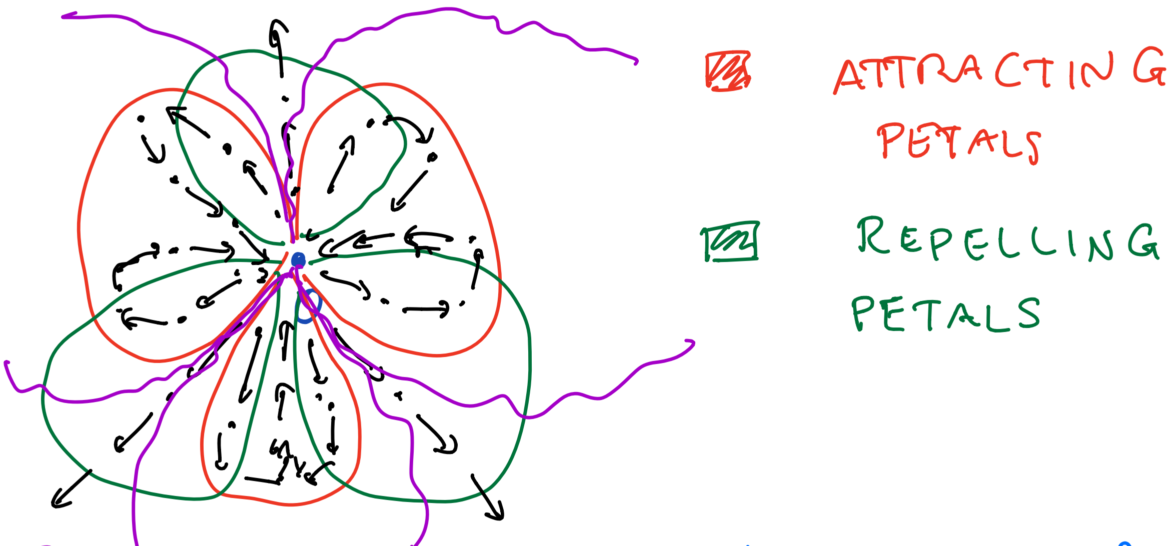
s.t. $\exists q : \lambda^q = 1$.

Let N, N' be nbds of 0 s.t. $f(N) = N'$.

Def.: an open $\mathcal{U}$ is an ATTRACTING PETAL if $\overline{\mathcal{U}} \subset N \cap N'$, $f(\overline{\mathcal{U}}) \subset \mathcal{U} \cup \{0\}$

$$\bigcap_{k \geq 0} f^k(\overline{\mathcal{U}}) = \{0\}$$

An open $\mathcal{U}$ is a REPELLING PETAL if it is an attracting petal for f^{-1}.



Rmk.: petals are not canonical

Cor.: attracting petals lie in the Fatou set; repelling petals intersects the Julia set.

Thm (Leau-Fatou flower theorem)

If $f(0) = 0$, $f(z) = z + a_{n+1} z^{n+1} + \ldots$

then there are n disjoint attracting petals $\mathcal{U}_1, \dots, \mathcal{U}_n$ and n disjoint repelling petals $\mathcal{U}_1', \dots, \mathcal{U}_n'$ s.t.

1. $\bigcup_{i=1}^{n} \mathcal{U}_i \cup \bigcup_{i=1}^{n} \mathcal{U}_i' \cup \{0\}$ is a nbd of 0

2. $\mathcal{U}_i \cap \mathcal{U}_j' = \emptyset$ unless $j = i, i+1 \pmod{n}$